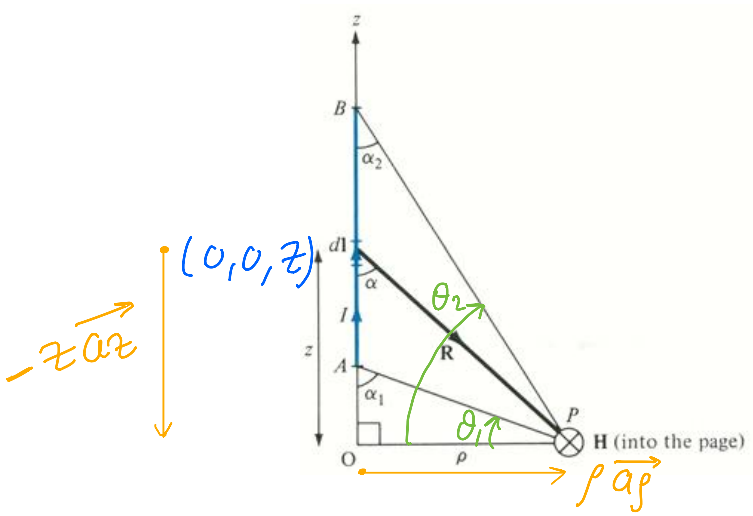
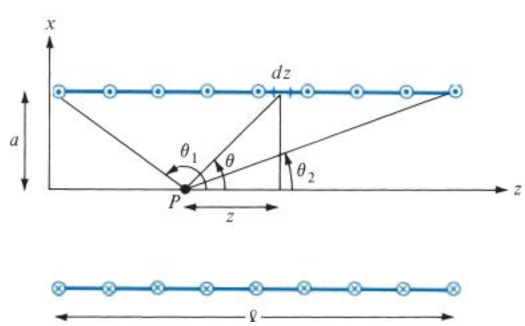
Princess Sumaya University for Technology
Communication Engineering Department
Electromagnetics I
Quiz 3

1. Determine the magnetic field ***H*** at point *P* due to a straight conductor of finite length *AB* and carrying current *I* as shown in the figure below.



2. A solenoid of length l and radius a consists of N turns and carries a current I as shown in the figure below. Find ***H*** along a point *P* along its axis. Then, find ***H*** at the center of the solenoid if $l \gg a$. Hence, calculate the self-inductance per unit length of an infinitely long solenoid.



Q1) $\vec{H} = \int \frac{I\vec{dl} \times \vec{R}}{4\pi|\vec{R}|^3}$

$I\vec{dl} = I\,dz\,\vec{a_z}$, $\vec{R} = -z\,\vec{a_z} + \rho\,\vec{a_\rho}$, $|\vec{R}| = \sqrt{z^2+\rho^2}$

$I\vec{dl} \times \vec{R} = I\,dz\,\vec{a_z} \times (\rho\,\vec{a_\rho} - z\,\vec{a_z})$

$= I\rho\,dz\,\underbrace{(\vec{a_z} \times \vec{a_\rho})}_{\vec{a_\phi}} - \cancel{I z\,dz\,(\vec{a_z} \times \vec{a_z})}^{\vec{0}} = I\rho\,dz\,\vec{a_\phi}$

$\therefore \vec{H} = \frac{1}{4\pi}\int_A^B \frac{I\rho\,dz}{(z^2+\rho^2)^{3/2}}\vec{a_\phi} = \frac{I\rho}{4\pi}\int_A^B \frac{dz}{(z^2+\rho^2)^{3/2}}\vec{a_\phi}$

Let $z = \rho\tan\theta \Rightarrow dz = \rho\sec^2\theta\,d\theta$

$\therefore \vec{H} = \frac{I\rho}{4\pi}\int_{\theta_1}^{\theta_2} \frac{\rho\sec^2\theta}{\rho^3\sec^3\theta}\,d\theta\,\vec{a_\phi} = \frac{I}{4\pi\rho}\int_{\theta_1}^{\theta_2} \cos\theta\,d\theta\,\vec{a_\phi}$

$= \frac{I}{4\pi\rho}\sin\theta\Big|_{\theta_1}^{\theta_2} \cdot \vec{a_\phi} = \frac{I}{4\pi\rho}(\sin\theta_2 - \sin\theta_1) \cdot \vec{a_\phi}$

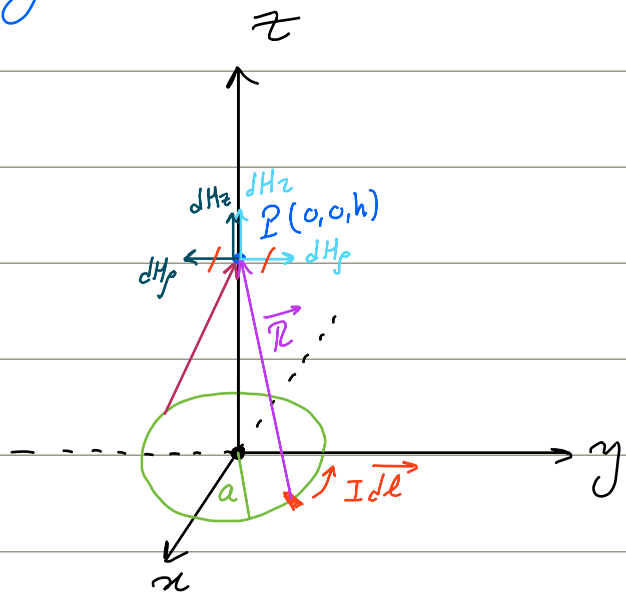
In terms of α: $\theta_2 = 90^\circ - \alpha_2$, $\theta_1 = 90^\circ - \alpha_1$

$\therefore \vec{H} = \frac{I}{4\pi\rho}\left[\sin(90^\circ-\alpha_2) - \sin(90^\circ-\alpha_1)\right]\vec{a_\phi}$

$\therefore \vec{H} = \frac{I}{4\pi\rho}(\cos\alpha_2 - \cos\alpha_1)\,\vec{a_\phi}$.

Q2)

Recall that $\vec{H}$ due to a current ring of radius a at point (0,0,z) is given by:



$$\vec{H} = \frac{I a^2}{2(z^2+a^2)^{3/2}} \vec{a_z}$$

But the solenoid consists of N turns and it has a length ℓ.

$$\therefore \ \vec{dH} = \frac{I a^2 \vec{dl}}{2(z^2+a^2)^{3/2}} \quad \text{where } \vec{dl} = \frac{N}{\ell} dz \, \vec{a_z}$$

$$\therefore \ \vec{dH} = \frac{I a^2 N}{2\ell(z^2+a^2)^{3/2}} dz \, \vec{a_z}$$

From the figure given in the question:

$$\tan\theta = \frac{a}{z} \Rightarrow z = a\cot\theta \Rightarrow dz = -a\csc^2\theta \, d\theta$$

Also from the figure: $\csc\theta = \dfrac{\sqrt{a^2+z^2}}{a}$

$$\therefore \ dz = -a \cdot \frac{a^2+z^2}{a^2} d\theta \Rightarrow dz = -\frac{a^2+z^2}{a} d\theta$$

$$\therefore \vec{dH} = \frac{Ia^2 N}{2\ell(a^2+z^2)^{3/2}} \cdot \overbrace{\frac{-(a^2+z^2)}{a} d\theta}^{dz} \ \vec{a_z}$$

$$\Rightarrow \vec{dH} = \frac{-IaN\,d\theta}{2\ell\sqrt{a^2+z^2}} \vec{a_z} \ , \quad \text{But } z = a\cot\theta$$

$$\therefore \vec{dH} = \frac{-IaN\,d\theta}{2\ell\sqrt{a^2+a^2\cot^2\theta}} \vec{a_z} = \frac{-IaN\,d\theta}{2\ell\sqrt{a^2(1+\cot^2\theta)}} \vec{a_z}$$

$$= \frac{-IaN\,d\theta}{2\ell\sqrt{a^2\csc^2\theta}} \vec{a_z} = \frac{-I\cancel{a}N\,d\theta}{2\ell\cancel{a}\csc\theta} \vec{a_z} = \frac{IN}{2\ell} \sin\theta\, d\theta \ \vec{a_z}$$

$$\vec{H} = \int_{\theta_1}^{\theta_2} \frac{-IN}{2\ell} \sin\theta\, d\theta \ \vec{a_z} = \frac{IN}{2\ell} \cos\theta \Big|_{\theta_1}^{\theta_2} \vec{a_z} = \frac{IN}{2\ell}\left[\cos\theta_2 - \cos\theta_1\right] \vec{a_z}$$

At the center: $\cos\theta_2 = -\cos\theta_1 = \dfrac{\ell/2}{\sqrt{(\ell/2)^2+a^2}}$

$$\therefore \vec{H} = \frac{IN}{2\ell}\left[\frac{\ell/2}{\sqrt{(\ell/2)^2+a^2}} + \frac{\ell/2}{\sqrt{(\ell/2)^2+a^2}}\right] \vec{a_z} = \frac{IN}{2\sqrt{\frac{\ell^2}{4}+a^2}} \vec{a_z}$$

If $\ell >> a$: $\vec{H} = \dfrac{IN}{2\ell}\left[\cos(0) - \cos(180^\circ)\right] \vec{a_z}$

$$= \frac{IN}{2\ell}(2)\vec{a_z} = \frac{IN}{\ell}\vec{a_z}.$$